Gent-McWilliams eddy Induced transport

The bolus velocity is defined as

$$\vec{u}^* = -(k S)_z$$

$$\nabla_z A = \nabla_s A - \frac{1}{H_z}\nabla_s z \frac{\partial A}{\partial s}$$

$$\frac{\partial}{\partial z} = \frac{1}{H_z}\frac{\partial}{\partial s}$$

and

$$S = -\frac{\nabla_z \rho}{\partial\rho/\partial z} = \frac{-\left(\nabla_s\rho - \frac{1}{H_z}\nabla_s z \frac{\partial\rho}{\partial z}\right)}{\frac{1}{H_z}\frac{\partial\rho}{\partial s}} = -H_z\frac{\nabla_s\rho}{\partial\rho/\partial s} + \nabla_s z$$

In s-coordinates, the bolus velocity becomes

$$\vec{u}^* = \frac{1}{H_z}\frac{\partial}{\partial s}\left[k\left(H_z\frac{\nabla_s\rho}{\partial\rho/\partial s} - \nabla_s z\right)\right]$$

so the components are

$$H_z u^* = \frac{\partial}{\partial s}\left[k\left(H_z\frac{\partial\rho/\partial\xi}{\partial\rho/\partial s} - \frac{\partial z}{\partial\xi}\right)\right]$$

$$H_z v^* = \frac{\partial}{\partial s}\left[k\left(H_z\frac{\partial\rho/\partial\eta}{\partial\rho/\partial s} - \frac{\partial z}{\partial\eta}\right)\right]$$

The continuity equation is

$$\frac{\partial}{\partial\xi}\left(\frac{H_z u^*}{n}\right) + \frac{\partial}{\partial\eta}\left(\frac{H_z v^*}{m}\right) + \frac{\partial}{\partial s}\left(\frac{H_z w^*}{mn}\right) = 0$$

so the vertical component is

$$\frac{H_z w^*}{mn} = \int_{-1}^{0}\left[\frac{\partial}{\partial\xi}\left(\frac{H_z u^*}{n}\right) + \frac{\partial}{\partial\eta}\left(\frac{H_z v^*}{m}\right)\right] ds$$