Isopycnic Fluxes

$$R(T) = \nabla_s^{3D} \cdot \frac{1}{H_\theta}\left[\nu H_\theta K \nabla_s^{3D} T\right]$$

$$K = \begin{bmatrix} I & S \\ S & S.S \end{bmatrix}$$

$$\nabla_s^{3D} = \left(\nabla_s , \frac{\partial}{\partial s}\right)$$

The small angle approximation can be writen as

$$R(T) = \nabla T . \vec{F}$$

where in (x,y,z) coordinates

$$S_x = - \frac{\partial\rho}{\partial x} \Big/ \frac{\partial\rho}{\partial z}$$

$$S_y = - \frac{\partial\rho}{\partial y} \Big/ \frac{\partial\rho}{\partial z}$$

$$-F^x = \nu_I\left(\frac{\partial T}{\partial x} - \frac{\partial T}{\partial z}\frac{\rho_x}{\rho_z}\right)$$

$$-F^y = \nu_I\left(\frac{\partial T}{\partial y} - \frac{\partial T}{\partial z}\frac{\rho_x}{\rho_z}\right)$$

$$-F^z = \nu_I\left(-\frac{\partial T}{\partial x}\frac{\rho_x}{\rho_z} - \frac{\partial T}{\partial y}\frac{\rho_y}{\rho_z} + \frac{\partial T}{\partial z}\frac{\rho_x^2+\rho_y^2}{\rho_z^2}\right) + \nu_D\frac{\partial T}{\partial z}$$

In s-coordinates, the diffusive fluxes become

$$\nabla_z A = \nabla_s A - \frac{1}{H_\theta}\nabla_s z\frac{\partial A}{\partial s}$$

$$\frac{\partial}{\partial z} = \frac{1}{H_\theta}\frac{\partial}{\partial s}$$

$$-F^\xi = \nu_I\left(\frac{\partial T}{\partial \xi} - \frac{1}{H_\theta}\frac{\partial z}{\partial \xi}\frac{\partial T}{\partial s} - \frac{1}{H_\theta}\frac{\partial T}{\partial s}S_\xi\right)$$

$$-F^\eta = \nu_I\left(\frac{\partial T}{\partial \eta} - \frac{1}{H_\theta}\frac{\partial z}{\partial \eta}\frac{\partial T}{\partial s} - \frac{1}{H_\theta}\frac{\partial T}{\partial s}S_\eta\right)$$

$$-F^s = \nu_I\left[-\left(\frac{\partial T}{\partial \xi} - \frac{1}{H_\theta}\frac{\partial z}{\partial \xi}\frac{\partial T}{\partial s}\right)S_\xi - \left(\frac{\partial T}{\partial \eta} - \frac{1}{H_\theta}\frac{\partial z}{\partial \eta}\frac{\partial T}{\partial s}\right)S_\eta + \frac{1}{H_\theta}\frac{\partial T}{\partial s}\left(S_\xi^2 + S_\eta^2\right)\right] + \nu_D\frac{1}{H_\theta}\frac{\partial T}{\partial s}$$

or

$$-F^\xi = \nu_I\frac{\partial T}{\partial \xi} - \frac{1}{H_\theta}\nu_I\left(\frac{\partial z}{\partial \xi} + S_\xi\right)\frac{\partial T}{\partial s}$$

$$-F^\eta = \nu_I\frac{\partial T}{\partial \eta} - \frac{1}{H_\theta}\nu_I\left(\frac{\partial z}{\partial \eta} + S_\eta\right)\frac{\partial T}{\partial s}$$

$$-F^s = \nu_D\frac{1}{H_\theta}\frac{\partial T}{\partial s} - F^\xi S_\xi - F^\eta S_\eta$$