$$\boxed{P_b = K_H\left[-\frac{g}{\rho_0}\frac{\partial\rho}{\partial z}\right] = -K_H N^2} \tag{9}$$

The dissipation of turbulent energy is

$$\boxed{\Xi_d = \frac{q^3}{B_1 l}} \tag{10}$$

where $l$ is the turbulent length scale.

Using the material derivative (1) in (2) yields, after multiplying by $\left(\frac{H_\theta}{mn}\right)$ and letting $A = \frac{q^2}{2}$

$$\frac{H_\theta}{mn}\frac{D}{Dt}\left(\frac{q^2}{2}\right) - \frac{H_\theta}{mn}\frac{\partial}{\partial z}\left[l_q S_q \frac{\partial}{\partial z}\left(\frac{q^2}{2}\right)\right] = \frac{H_\theta}{mn}\left[P_s + P_b + \Xi_d\right]$$

and expanding using (1)

$$\boxed{\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{H_\theta q^2}{mn}\right) + \frac{1}{2}\frac{\partial}{\partial \xi}\left(\frac{H_\theta u q^2}{n}\right) + \frac{1}{2}\frac{\partial}{\partial \eta}\left(\frac{H_\theta v q^2}{m}\right) + \frac{1}{2}\frac{\partial}{\partial s}\left(\frac{H_\theta \Omega q^2}{mn}\right) \\ - \frac{1}{2}\frac{H_\theta}{mn}\frac{\partial}{\partial z}\left[l q S_q \frac{\partial}{\partial z}(q^2)\right] = \frac{H_\theta}{mn}\left[P_s + P_b - \Xi_d\right]\end{aligned}} \tag{11}$$

let

$$\boxed{K_Q = l q S_Q + \nu} \qquad \boxed{S_Q = k S_M} \tag{12}$$

and using $\frac{\partial}{\partial z} = \frac{1}{H_\theta}\frac{\partial}{\partial s}$ and adding horizontal dissipation $D_q$ gives

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{H_\theta q^2}{mn}\right) + \frac{1}{2}\frac{\partial}{\partial \xi}\left(\frac{H_\theta u q^2}{n}\right) + \frac{1}{2}\frac{\partial}{\partial \eta}\left(\frac{H_\theta v q^2}{m}\right) + \frac{1}{2}\frac{\partial}{\partial s}\left(\frac{H_\theta \Omega q^2}{mn}\right)$$

$$- \frac{1}{2}\frac{H_\theta}{mn}\frac{1}{\cancel{H_\theta}}\frac{\partial}{\partial s}\left[\frac{K_Q}{H_\theta}\frac{\partial}{\partial s}(q^2)\right] = \frac{H_\theta}{mn}\left[P_s - P_b + \Xi_d\right] + \frac{H_\theta}{mn} D_q$$

where $D_q$ is the Laplacian or biharmonic operator.