5.- Vertically Integrated Equations

The depth-averaged integral for any fluid property (scalar or vector) is defined as

$$\boxed{\bar{A} = \frac{1}{D}\int_{-1}^{0} (H_\theta A)\, ds} \qquad (5.1)$$

where

$$\boxed{D \equiv \zeta(\xi,\eta,t) + h(\xi,\eta)} \qquad (5.2)$$

is the total, time-dependent depth of the water column

(a) Momentum Equations

The vertically integrated equation in the ξ-direction is obtained by integrating (4.7) and using (5.1) and boundary conditions (4.24) and (4.27).

Notice that,

$$\int_{-1}^{0} \frac{\partial}{\partial t}\left(\frac{H_\theta u}{mn}\right) ds = \frac{\partial}{\partial t}\left(\frac{D\bar{u}}{mn}\right) \; ; \qquad D\bar{u} = \int_{-1}^{0} H_\theta u\, ds$$

$$\int_{-1}^{0} \frac{\partial}{\partial \xi}\left(\frac{H_\theta uu}{n}\right) ds = \frac{\partial}{\partial \xi}\left(\frac{D\overline{uu}}{n}\right) \; ; \qquad D\overline{uu} = \int_{-1}^{0} H_\theta uu\, ds$$

$$\int_{-1}^{0} \frac{\partial}{\partial \eta}\left(\frac{H_\theta vu}{m}\right) ds = \frac{\partial}{\partial \eta}\left(\frac{D\overline{vu}}{m}\right) \; ; \qquad D\overline{vu} = \int_{-1}^{0} H_\theta vu\, ds$$

$$\int_{-1}^{0} \frac{\partial}{\partial s}\left[\left(\frac{K_M}{H_\theta mn}\right)\frac{\partial u}{\partial s}\right] ds = \frac{1}{mn}\left[\frac{K_M}{H_\theta}\frac{\partial u}{\partial s}\bigg|_{s=0} - \frac{K_M}{H_\theta}\frac{\partial u}{\partial s}\bigg|_{s=-1}\right]$$

$$= \frac{1}{mn}\left(\tau_s^{\xi} - \tau_b^{\xi}\right)$$

$$\int_{-1}^{0} H_\theta \frac{\partial p_T}{\partial \xi}\, ds = D\frac{\partial}{\partial \xi}(\bar{p}_T) \; ; \qquad D\bar{p}_T = \int_{-1}^{0} H_\theta p_T\, ds$$