Expanding the integral gives

$$\nabla_z \phi = \frac{g}{\rho_0}\int_s^0 \frac{\partial z}{\partial s}\nabla_s \rho \, ds + \frac{g}{\rho_0}\int_s^0 \rho \underbrace{\nabla_s\left(\frac{\partial z}{\partial s}\right)}_{\frac{\partial}{\partial s}\nabla_s z} ds + \frac{g}{\rho_0}\rho \nabla_s z$$

Integrating by parts the second integral yields

$$\frac{g}{\rho_0}\int_s^0 \rho \frac{\partial}{\partial s}(\nabla_s z)\, dz = \frac{g}{\rho_0}\Big[\rho \nabla_s z\Big]_{s'=s}^{s'=0} - \frac{g}{\rho_0}\int_s^0 \frac{\partial \rho}{\partial s}\nabla_s z \, ds$$

$\int u\,dv = uv - \int v\,du$

$$= \frac{g}{\rho_0}\underbrace{\rho(s=0)}_{\rho_{surface}}\nabla_s \zeta - \frac{g}{\rho_0}\rho\nabla_s z - \frac{g}{\rho_0}\int_s^0 \frac{\partial \rho}{\partial s}\nabla_s z \, ds$$

$u = \rho \qquad dv = \frac{\partial}{\partial s}(\nabla_s z)\,ds$

$du = \frac{\partial \rho}{\partial s}ds \qquad v = \nabla_s z$

Therefore, $\nabla_z \phi$ becomes

$$\nabla_z \phi = \frac{g}{\rho_0}\left[\int_s^0 \frac{\partial z}{\partial s}\nabla_s \rho \, ds + \rho_{surface}\nabla_s \zeta - \cancel{\rho\nabla_s z} - \int_s^0 \frac{\partial \rho}{\partial s}\nabla_s z\, ds + \cancel{\rho \nabla_s z}\right]$$

Combining the integrals gives

$$\boxed{\nabla_z \phi = \frac{g}{\rho_0}\left[\rho_{surface}\nabla_s \zeta + \int_s^0 \left(\frac{\partial z}{\partial s}\nabla_s \rho - \frac{\partial \rho}{\partial s}\nabla_s z\right) ds\right.} \qquad (4.15)$$

Now, the $\nabla_z \zeta$ component in (4.12) becomes, after using (2.9)

$$\nabla_z \zeta = \nabla_s \zeta - \frac{1}{H_0}\underbrace{\cancel{\frac{\partial \zeta}{\partial s}}}_{0}\nabla_s z$$

$$\boxed{\nabla_z \zeta = \nabla_s \zeta} \qquad (4.16)$$