① Pressure Gradient term in (x, y, s, t) coordinates

The hydrostatic equation for total pressure, p_T, and total density, $\rho_0+\rho$ is

$$\boxed{\frac{\partial}{\partial z}(p_T) = -\frac{g}{\rho_0}(\rho_0+\rho)} \qquad (4.9)$$

integrating over the range of z gives

$$\int_z^{\zeta(x,y,t)} \frac{\partial}{\partial z}(p_T)\,dz = -\frac{g}{\rho_0}\int_z^{\zeta(x,y,t)} (\rho_0+\rho)\,dz$$

$$\underbrace{\cancel{p_T(z=\zeta)}}_{0} - p_T(x,y,z,t) = -\frac{g}{\rho_0}\int_z^{\zeta(x,y,t)} (\rho_0+\rho)\,dz$$

$$p_T = \frac{g}{\rho_0}\int_z^{\zeta(x,y,t)} (\rho_0+\rho)\,dz$$

Therefore

$$\boxed{p_T = \frac{g}{\cancel{\rho_0}}\left[\cancel{\rho_0}(\zeta - z)\right] + \frac{g}{\rho_0}\int_z^{\zeta(x,y,t)} \rho\,dz} \qquad (4.10)$$

define dynamic pressure, ϕ such that

$$\boxed{\frac{\partial\phi}{\partial z} = -\frac{g}{\rho_0}\rho} \qquad \boxed{\phi = \frac{g}{\rho_0}\int_z^{\zeta(x,y,t)} \rho\,dz} \qquad \phi = 0 \;@\; z = \zeta(x,y,t) \qquad (4.11)$$

Taking the horizontal gradient of (4.10) and using (3.9) gives

$$\nabla_z p_T = g\,\nabla_z(\zeta - z) + \nabla_z\phi$$

$$= g\,\nabla_z\zeta - \underbrace{\cancel{g\,\nabla_z z}}_{0} + \nabla_z\phi$$