4. Governing Equations in (ξ, η, σ, t) coordinates

(a) An alternative form of the momentum equation (1.1) can be obtained by using the vector identity

$$\vec{\omega} \times \vec{V} = (\vec{V} \cdot \nabla)\vec{V} - \nabla\left(\tfrac{1}{2} V \cdot V\right) \qquad (4.1)$$

Therefore

$$\frac{\partial \vec{V}}{\partial t} + (f+\delta)\hat{k} \times V + w\frac{\partial V}{\partial z} = -\nabla_z\left(P + \tfrac{1}{2} V \cdot V\right) + \frac{\partial}{\partial z}\left(K_m \frac{\partial \vec{V}}{\partial z}\right) + \mathcal{D}_V + F_V \qquad (4.2)$$

the ξ-component of (4.2) is then $\hat{k} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ u & v & 0 \end{vmatrix} = -v\hat{i} + u\hat{j}$

$$\frac{\partial u}{\partial t} - \left\{ f + mn\left[\frac{\partial}{\partial \xi}\left(\frac{v}{n}\right) - \frac{\partial}{\partial \eta}\left(\frac{u}{m}\right)\right]\right\} v + \dot{\sigma}\frac{\partial u}{\partial \sigma} + m\frac{\partial}{\partial \xi}\left(P + \tfrac{1}{2}u^2 + \tfrac{1}{2}v^2\right) = F_\xi$$

the rotation term can be expanded as

$$-mn\left[\frac{\partial}{\partial \xi}\left(\frac{v}{n}\right) - \frac{\partial}{\partial \eta}\left(\frac{u}{m}\right)\right] v = -mn\left[v\frac{\partial}{\partial \xi}\left(\frac{1}{n}\right) - u\frac{\partial}{\partial \eta}\left(\frac{1}{m}\right)\right] v - mn\left[\frac{1}{n}\frac{\partial v}{\partial \xi} - \frac{1}{m}\frac{\partial u}{\partial \eta}\right] v$$

$$-mv\frac{\partial v}{\partial \xi} + nv\frac{\partial u}{\partial \eta}$$

the pressure gradient term can be expanded as

$$m\frac{\partial}{\partial \xi}\left(P + \tfrac{1}{2}u^2 + \tfrac{1}{2}v^2\right) = m\frac{\partial}{\partial \xi}(P) + mu\frac{\partial u}{\partial \xi} + mv\frac{\partial v}{\partial \xi}$$

the ξ-component of (4.2) becomes

$$\frac{\partial u}{\partial t} + mu\frac{\partial u}{\partial \xi} + nv\frac{\partial u}{\partial \eta} + \dot{\sigma}\frac{\partial u}{\partial \sigma} - \left\{ f + mn\left[v\frac{\partial}{\partial \xi}\left(\frac{1}{n}\right) - u\frac{\partial}{\partial \eta}\left(\frac{1}{m}\right)\right]\right\} v + m\frac{\partial P}{\partial \xi} + mv\frac{\partial v}{\partial \xi} - mv\frac{\partial v}{\partial \xi} = F_\xi$$

(4.3)