(e) Continuity Equation

starting from the continuity equation in (x,y,z,t) coordinates,

$$\nabla_z \cdot \vec{V} + \frac{\partial w}{\partial z} = 0$$

using identities (2.9) and (2.10) and

$$\boxed{w = \frac{\partial z}{\partial t} + \vec{V}\cdot\nabla_s z + \Omega H_\theta} \qquad (2.11)$$

gives

$$\left[\nabla_s - \frac{1}{H_\theta}\nabla_s z \frac{\partial}{\partial s}\right]\cdot\vec{V} + \frac{1}{H_\theta}\frac{\partial}{\partial s}\left[\frac{\partial z}{\partial t} + \vec{V}\cdot\nabla_s z + \Omega H_\theta\right] = 0$$

$$\nabla_s\cdot\vec{V} \underbrace{- \frac{1}{H_\theta}\nabla_s z\cdot\frac{\partial V}{\partial s} + \frac{1}{H_\theta}\frac{\partial}{\partial s}(\vec{V}\cdot\nabla_s z)}_{\frac{1}{H_\theta}\vec{V}\cdot\frac{\partial}{\partial s}(\nabla_s z)\;\to\;\frac{1}{H_\theta}\vec{V}\cdot\nabla_s H_\theta} + \frac{1}{H_\theta}\frac{\partial}{\partial t}\Big(\underbrace{\frac{\partial z}{\partial s}}_{H_\theta}\Big) + \frac{1}{H_\theta}\frac{\partial}{\partial s}(\Omega H_\theta) = 0$$

$\frac{\partial}{\partial s}(\vec{A}\cdot\vec{B}) = \vec{A}\cdot\frac{\partial\vec{B}}{\partial s} + \vec{B}\frac{\partial\vec{A}}{\partial s}$

$\vec{A} = \vec{V}$

$\vec{B} = \nabla z$

multiplying by H_θ gives

$$\frac{\partial H_\theta}{\partial t} + \underbrace{H_\theta(\nabla_s\cdot\vec{V}) + \vec{V}\cdot(\nabla_s H_\theta)}_{\nabla_s\cdot(H_\theta\vec{V})} + \frac{\partial}{\partial s}(\Omega H_\theta) = 0$$

$\nabla\cdot(S\vec{A}) = (\nabla S)\cdot\vec{A} + S(\nabla\cdot\vec{A})$

resulting in the continuity equation

$$\boxed{\frac{\partial H_\theta}{\partial t} + \nabla_s\cdot(H_\theta\vec{V}) + \frac{\partial}{\partial s}(\Omega H_\theta) = 0} \qquad \text{Flux Form} \qquad (2.12)$$