(b) Material Derivative in x, y, s, t coordinates

The material derivative in (x,y,s,t) coordinates is

$$\left(\frac{DA}{Dt}\right)_s = \left(\frac{\partial A}{\partial t}\right)_s + \left(\vec{V}\cdot\nabla_s\right)A + \Omega\frac{\partial A}{\partial s} = 0 \qquad \text{advective form} \qquad (2.7)$$

where again A is any fluid property (scalar or vector), and Ω is the "omega" vertical velocity.

(c) Vertical velocity in x, y, s, t coordinates

Equating (1.6) and (2.7) and the using (2.1) gives

$$\left(\frac{Dz}{Dt}\right)_s = \left(\frac{Dz}{Dt}\right)_z$$

$$\left(\frac{\partial z}{\partial t}\right)_s + \vec{V}\cdot\nabla_s z + \Omega\cancelto{H_\theta}{\frac{\partial z}{\partial s}} = \cancelto{0}{\left(\frac{\partial z}{\partial t}\right)_z} + \vec{V}\cdot\cancelto{0}{\nabla_z z} + w\cancelto{1}{\frac{\partial z}{\partial z}}$$

giving

$$\Omega H_\theta = \left[w - \left(\frac{\partial z}{\partial t}\right)_s - \vec{V}\cdot\nabla_s z\right]$$

From (2.1) we get $\frac{\partial z}{\partial t} = (1+s)\frac{\partial s}{\partial t}$

$\Omega : \frac{1}{s}$

$\Omega H_\theta : \frac{m}{s}$

$$\boxed{\Omega = \frac{1}{H_\theta}\left[w - (1+s)\frac{\partial s}{\partial t} - \vec{V}\cdot\nabla_s z\right]} \qquad (2.8)$$

(d) Usefull tranoformation operators

$$\boxed{\nabla_z A = \nabla_s A - \frac{1}{H_\theta}\nabla_s z\,\frac{\partial A}{\partial s}} \qquad (2.9)$$

$$\boxed{\frac{\partial A}{\partial z} = \frac{1}{H_\theta}\frac{\partial A}{\partial s}} \qquad (2.10)$$

where A is any fluid property (scalar or vector)