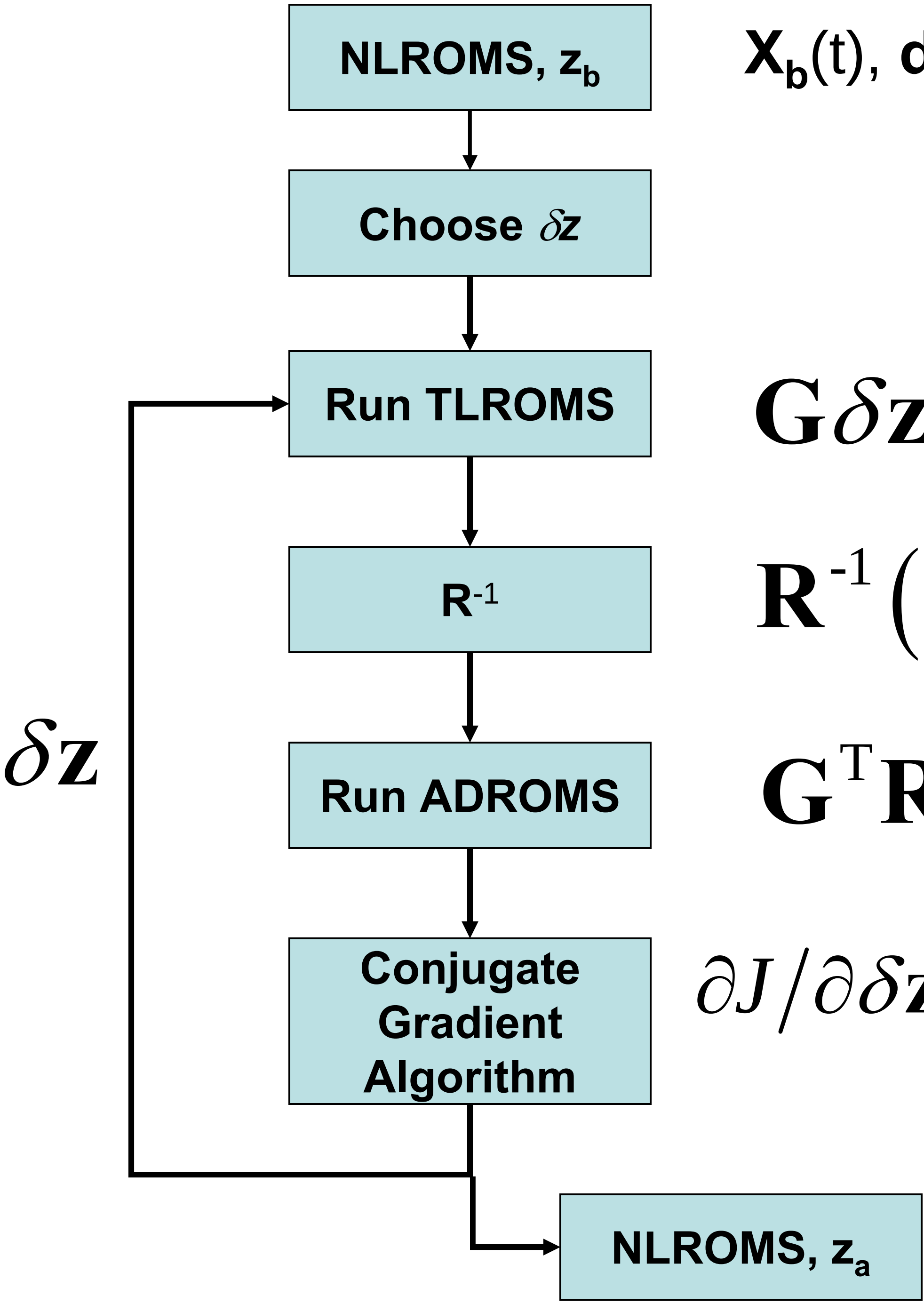
Tutorial 2:
Multiple Outer-Loops

Primal 4D-Var Algorithm (I4D-Var)



$X_b(t)$, **d**

$$\mathbf{G}\delta\mathbf{z}, \quad \left(\mathbf{G}\delta\mathbf{z}-\mathbf{d}\right), \quad J$$

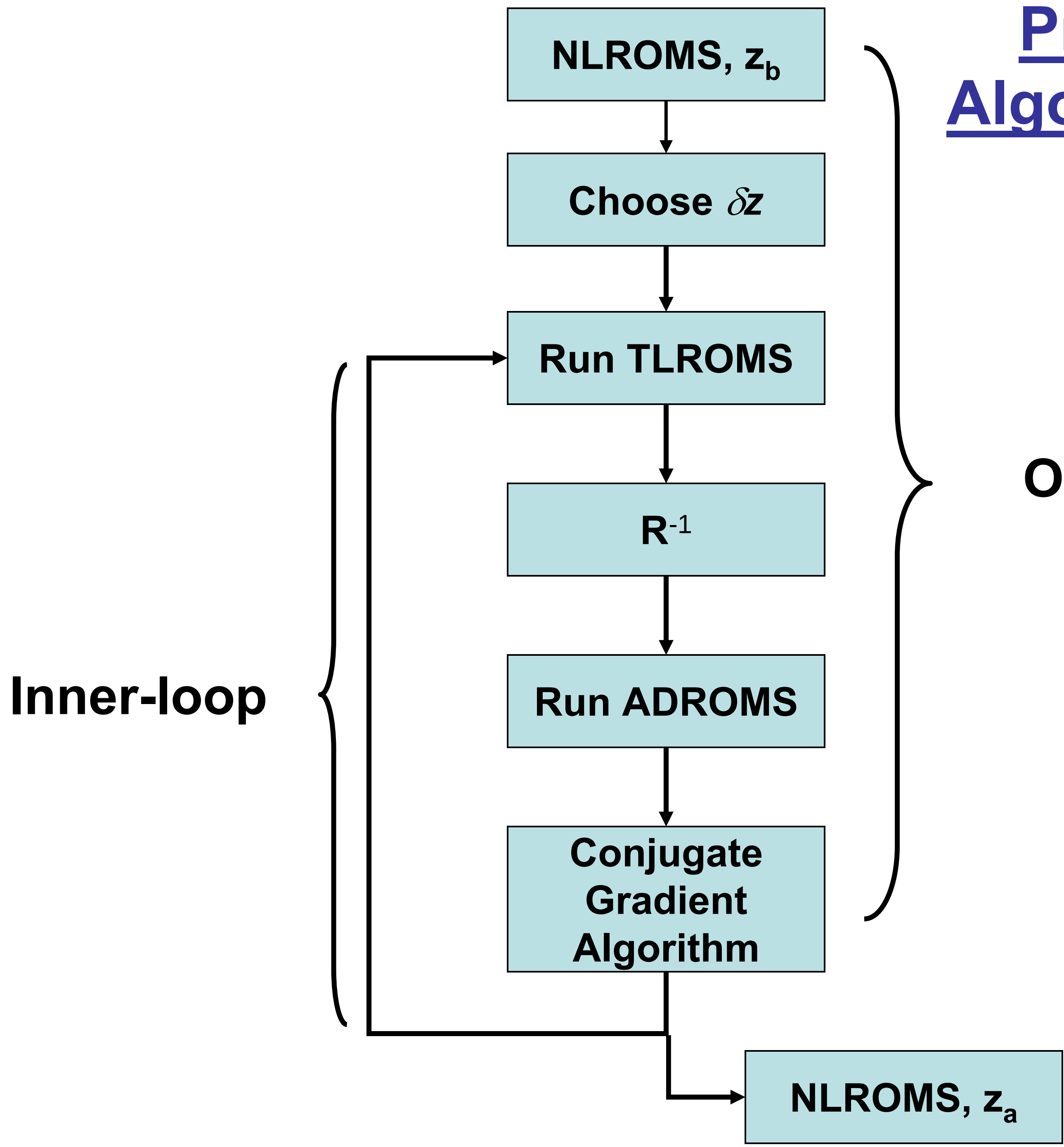
$$\mathbf{R}^{-1}\left(\mathbf{G}\delta\mathbf{z}-\mathbf{d}\right)$$

$$\mathbf{G}^{\mathrm{T}}\mathbf{R}^{-1}\left(\mathbf{G}\delta\mathbf{z}-\mathbf{d}\right)$$

$$\partial J/\partial\delta\mathbf{z} = \mathbf{D}^{-1}\delta\mathbf{z} + \mathbf{G}^{\mathrm{T}}\mathbf{R}^{-1}\left(\mathbf{G}\delta\mathbf{z}-\mathbf{d}\right)$$

$X_a(t)$

Primal 4D-Var Algorithm (I4D-Var)

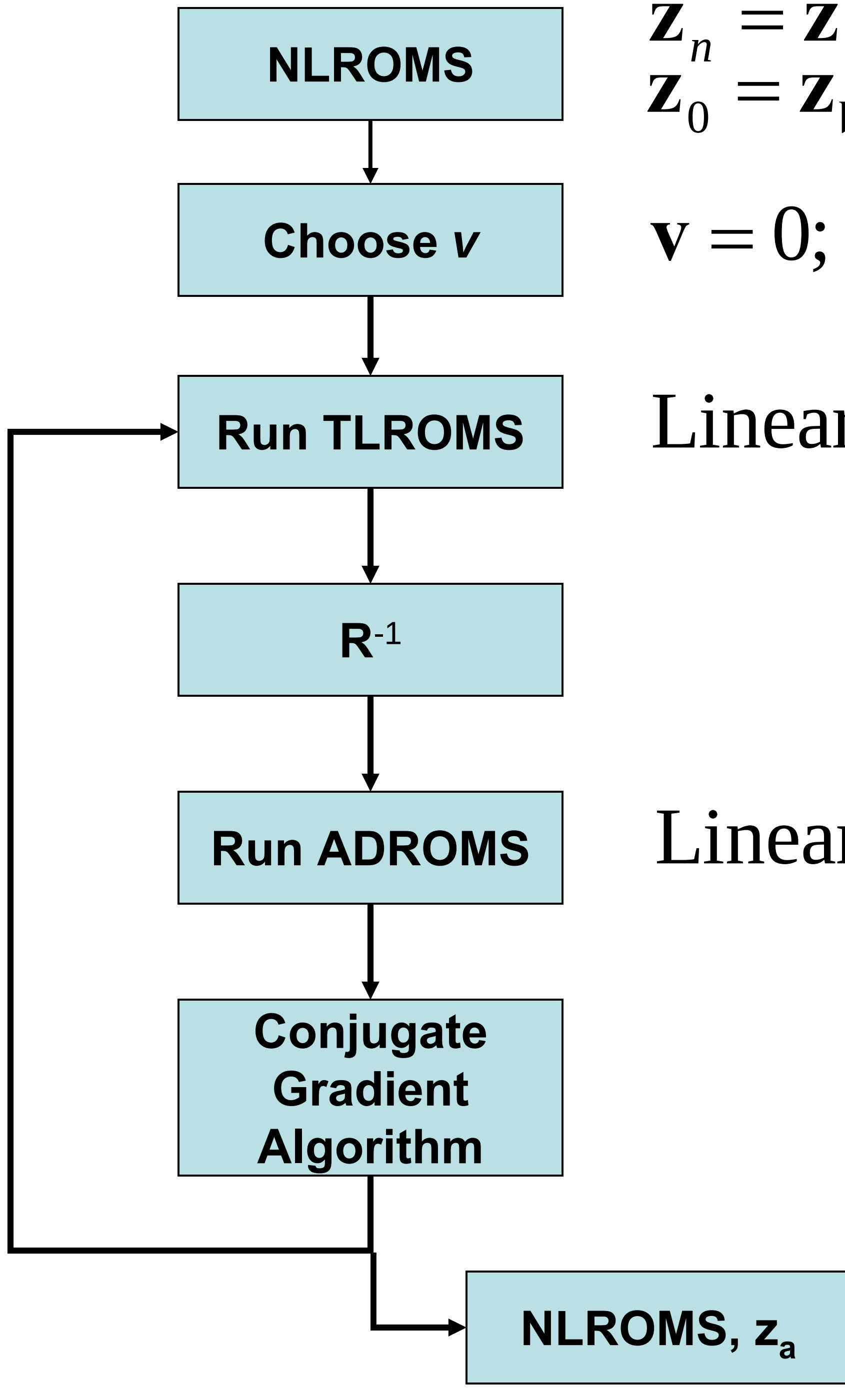


Multiple Outer-Loops

- It is often advantageous to perform more than one outer-loop.
- During each outer-loop the NL ROMS solution is updated using the increments $\delta\mathbf{z}$ from the last inner-loop of the previous outer-loop.
- The updated NL ROMS solution is that about which TL ROMS and AD ROMS are linearized during the current outer-loop.
- Equivalent to solving a sequence of linear least-squares minimization problems.
- Can potentially help in identifying the global minimum of J_{NL} although not proven.

$$J_{NL}(\mathbf{x}) = \frac{1}{2}(\mathbf{z}-\mathbf{z_b})^{\mathrm{T}}\mathbf{D}^{-1}(\mathbf{z}-\mathbf{z_b}) + \frac{1}{2}(\mathbf{y}-H(\mathbf{x}))^{\mathrm{T}}\mathbf{R}^{-1}(\mathbf{y}-H(\mathbf{x}))$$

Outer-loop, n
Inner-loop, m
1≤m≤M



$$\mathbf{z}_n = \mathbf{z}_{n-1} + \delta\mathbf{z}_{n-1}$$

$$\mathbf{z}_0 = \mathbf{z}_\mathbf{b}; \quad \delta\mathbf{z}_0 = 0$$

$$\mathbf{v} = 0; \quad \mathbf{v} = \mathbf{D}^{-1/2}\delta\mathbf{z}$$

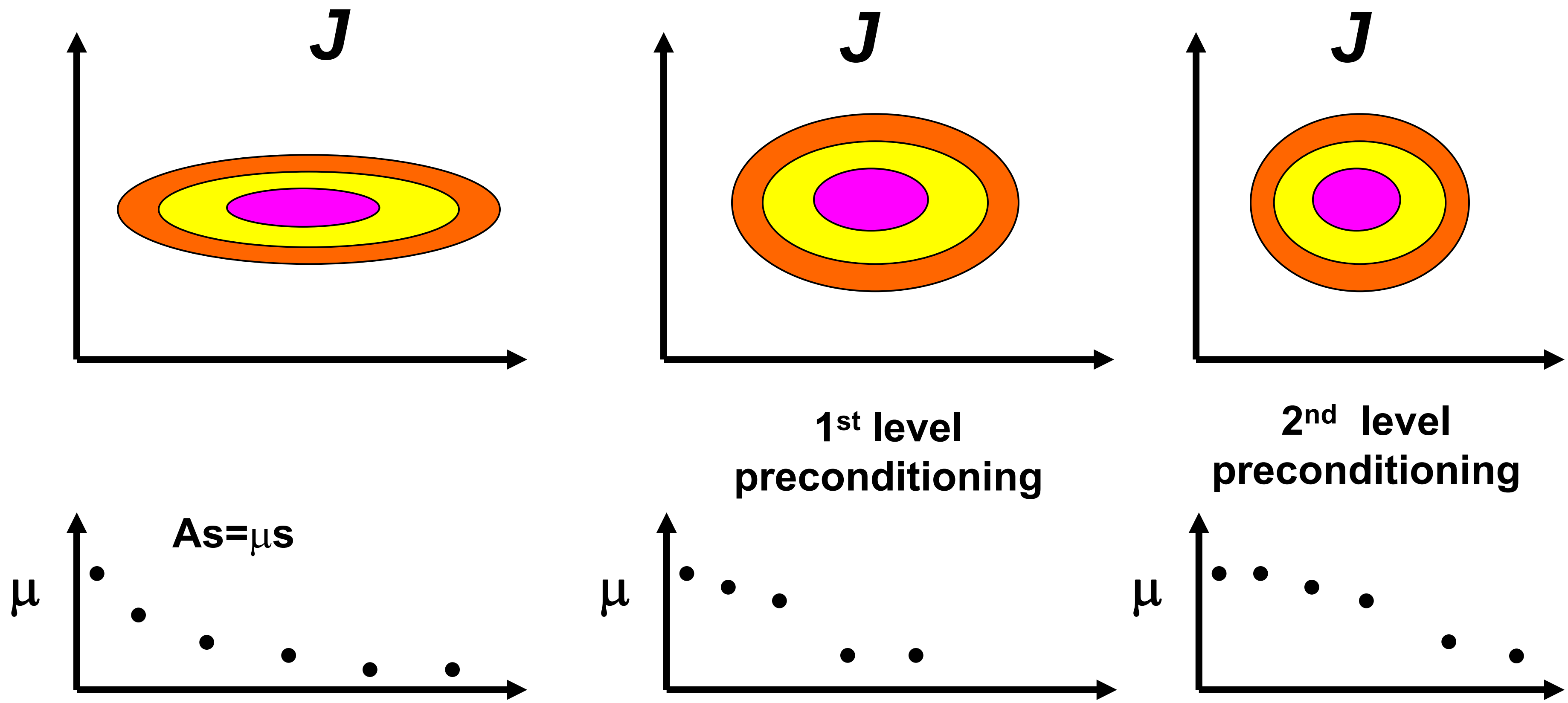
Linearize about $\mathbf{x}_n(t)$

Linearize about $\mathbf{x}_n(t)$

Implementation in ROMS 4D-Var

- **Number of inner-loops, *m*, and outer-loops, *n*, are controlled by *Ninner* and *Nouter* in the ocean.in file.**
- **Second level preconditioning option available using info gleaned about the shape of *J* (J_{NL}*?*) from previous outer-loops:**
  - **spectral & Ritz preconditioning (Tshimanga et al, 2008)**
  - **s4dvar.in**
    - **Lprecond (T (on) or F (off))**
    - **Lritz (T (ritz) or F (spectral))**
    - **NritzEV (number of vectors to use)**

Preconditioning



2^{nd} Level Preconditioning

Recall 1^{st} level preconditioning via: $\mathbf{v} = \mathbf{D}^{-1/2}\delta\mathbf{z}$

2^{nd} level preconditioning proceeds via: $\mathbf{u} = \mathbf{U}_n^{-1}\mathbf{v}$

(n= outer-loop #)

Recall: $\tilde{\mathbf{K}}_m = \mathbf{K_b}\boldsymbol{\Sigma}\mathbf{C}^{1/2}\mathbf{V}_m\mathbf{T}_m^{-1}\mathbf{V}_m^{\mathrm{T}}\mathbf{C}^{\mathrm{T}/2}\boldsymbol{\Sigma}^{\mathrm{T}}\mathbf{K}_{\mathbf{b}}^{\mathrm{T}}\mathbf{G}^{T}\mathbf{R}^{-1}$

(m= # of inner-loops)

Following Tshimanga et al (2008):

Spectral (Lritz=F)

$$\mathbf{U}_n^{-1} = \prod_{i=m}^{1}\left(\mathbf{I} - \left(1-\theta_i^{1/2}\right)\hat{\mathbf{y}}_i\hat{\mathbf{y}}_i^{\mathrm{T}}\right)$$

(correction for errors in y_i)

Ritz (Lritz=T)

$$\mathbf{U}_n^{-1} = \prod_{i=m}^{1}\left(\mathbf{I} - \left(1-\theta_i^{1/2}\right)\hat{\mathbf{y}}_i\hat{\mathbf{y}}_i^{\mathrm{T}} \boxed{-\theta_i^{-1}\left(\mathbf{e}_{\mathbf{m}}^{\mathrm{T}}\mathbf{y}_i\right)\gamma_m\hat{\mathbf{y}}_i\mathbf{q}_{m+1}^{\mathrm{T}}}\right)$$

where $\left(\theta_i, \mathbf{y}_i\right)$ **are eigenpairs of** $\mathbf{T}_m$ **and** $\hat{\mathbf{y}}_i = \mathbf{V}_m\mathbf{y}_i$

2^{nd} Level Preconditioning

Finally: $$\mathbf{v}_n = \prod_{j=1}^{n-1} \mathbf{U}_j \mathbf{u}_j$$

and: $$\delta \mathbf{z}_n = \mathbf{D}_n^{1/2} \mathbf{v}_n = \left(\mathbf{K_b}\right)_n \mathbf{\Sigma C}^{1/2} \mathbf{v}_n$$

ROMS CCS example:

Ritz #	0	1	2	3	4	8
J	9173	3131	1877	1855	1833	1972

4X25

(Moore et al, 2010b)

References

- Moore, A.M., H.G. Arango, G. Broquet, B.S. Powell, J. Zavala-Garay and A.T. Weaver, 2010a: The Regional Ocean Modeling System (ROMS) 4-dimensional data assimilation systems: Part I – System overview. *Ocean Modelling*, Submitted.
- Moore, A.M., H.G. Arango, G. Broquet, C.. Edwards, M. Veneziani, B.S. Powell, D. Foley, J. Doyle, D. Costa and P. Robinson, 2010b: The Regional Ocean Modeling System (ROMS) 4-dimensional data assimilation systems: Part II – Performance and application to the California Current System. *Ocean Modelling*, Submitted.
- Tshimanga, J., S. Gratton, A.T. Weaver and A. Sartenaer, 2008: Limited-memory preconditioners with application to incremental variational data assimilation. *Q. J. R. Meteorol. Soc*., **134**, 751-769.