Minimizing spurious diapycnal mixing induced by horizontal tracer advection

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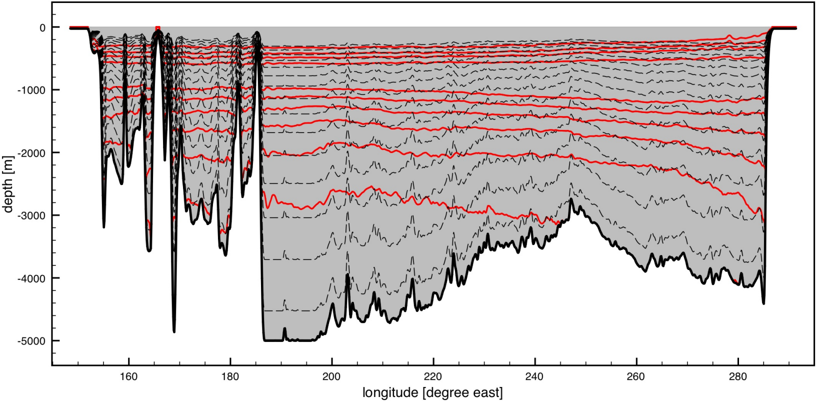
Spurious diapycnal mixing, why?

- Strong horizontal mixing along iso-sigma surfaces can lead to spurious diapycnal fluxes of heat and salt (and hence density) when it is applied in regions with sloping isopycnals.



Vertical section in the southern Pacific

Outline

1. Introduction and motivations
2. Neutral diffusion in ocean models
3. Advection and rotated diffusion coupling
4. Preliminary numerical test
5. Conclusion and perspectives

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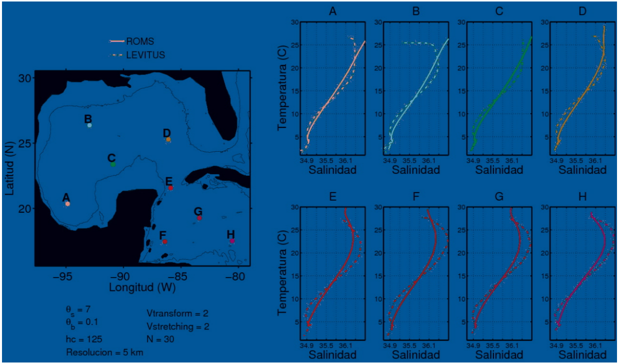
Identification of the problem

- The implicit diffusion associated to *odd-order* advection schemes may be large enough to produce excessive diapycnal mixing.

⟶ South West Pacific Region (Marchesiello *et. al*, 2009 ; Couvelard *et. al*, 2008)

⟶ Gulf of Mexico



Courtesy of Lorena Guerrero and Julio Sheinbaum

Why do we like the third order upstream advection scheme (U3H / UP3) ?

- *odd-ordered* scheme = next higher ordered scheme plus a dissipation term

$$\widetilde{q}_{i+\frac{1}{2}} = \underbrace{\frac{7(q_i + q_{i+1}) - (q_{i-1} + q_{i+2})}{12}}_{4^{\text{th}}\text{ order advection}} + \text{Sign}(\text{u}_{\text{i}+\frac{1}{2}}) \underbrace{\frac{3(q_{i+1} - q_i) - (q_{i+2} - q_{i-1})}{12}}_{\text{implicit diffusion}}$$

- Vertical advection not considered in this talk (small inherent diffusion)

3rd upstream-biased, parabolic interpolation

- Good compromise between accuracy, stability, simplicity and computational efficiency
- Provides a resolution dependent and scale-selective parameterization of dissipation

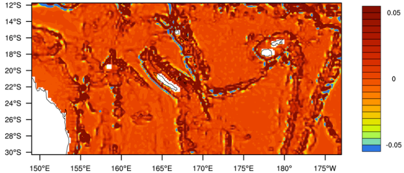
$\Rightarrow$ **However the adiabatic property of advection is broken**

Diapycnal component of the implicit diffusion

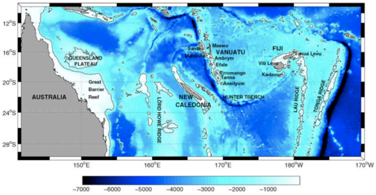
$\longrightarrow$ Online computation :

$$\Phi_\perp = \left((\vec{F}_{\mathrm{C4}} - \vec{F}_{\mathrm{UP3}}) \cdot \hat{\rho} \right) \rho_3 \text{ with } \hat{\rho} = (\rho_1, \rho_2, \rho_3)^t = \frac{\vec{\nabla}\rho^{(\mathrm{ad})}}{\|\vec{\nabla}\rho^{(\mathrm{ad})}\|}$$

$$\frac{1}{H} \int_H^{h_{bl}} \Phi_\perp dz$$

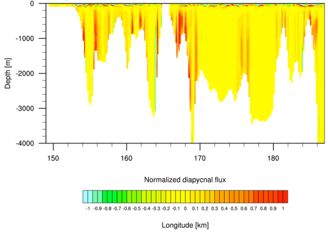


Topography of the Southwest Pacific region



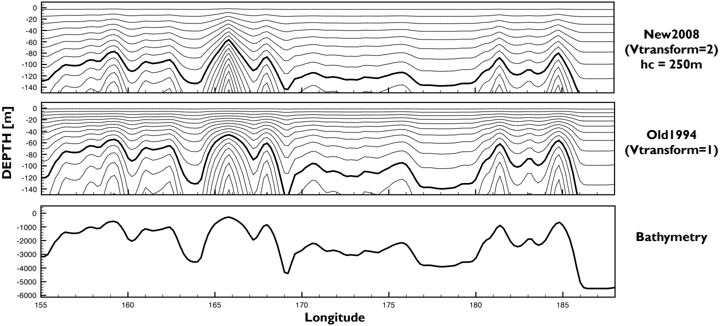
(Marchesiello *et al.*, 2009)

$\Phi_\perp$, vertical section at 21^o S



A first remedy : tuning of the vertical grid (and bathymetry smoothing)

→ Take advantage of the absence of restriction $h_c < h_{\min}$ with the new ("$z - \sigma$") coordinates ⇒ flattening of the near surface σ-levels

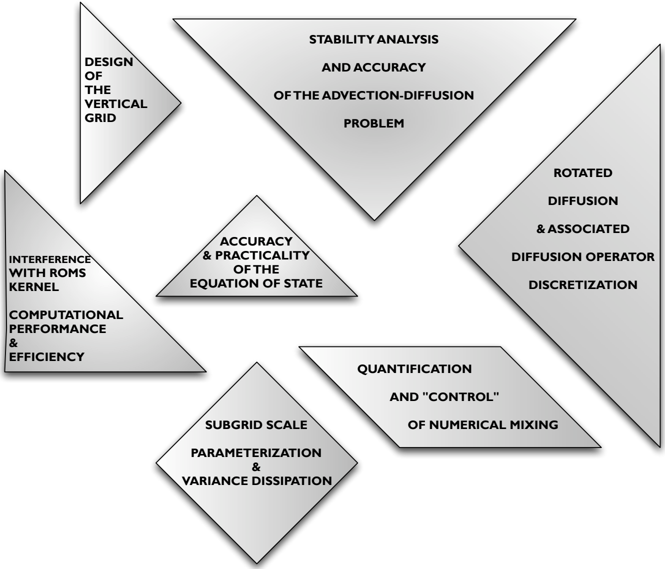


Shchepetkin's law : $h_c \approx 1.5 \times H_{\text{cline}}$

→ Bathymetry smoothing : see scaling of diapycnal mixing in Marchesiello *et. al*, 2009

Putting the Pieces of the Puzzle Together

→ The aim is not to suppress diffusion but to keep it "under control"



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Neutral diffusion in ocean models

$\longrightarrow$ Isopycnal mixing via Coordinate Rotation (Redi, 1982)

$$\mathcal{D} = \nabla \cdot (K_h \mathbf{K} \cdot \nabla q)$$

$\mathbf{K}$ = diffusion (Redi) tensor

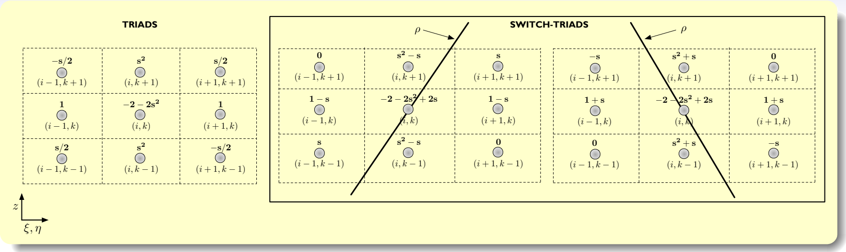
$\longrightarrow$ Example in 2D ($\xi - z$)

$$\mathcal{D} = K_h \partial_\xi^2 q \qquad \longrightarrow \qquad \frac{K_h}{1 + \left[\frac{\partial_\xi \rho}{\partial_z \rho}\right]^2} \left(\partial_\xi - \left[\frac{\partial_\xi \rho}{\partial_z \rho}\right] \partial_z \right)^2 q$$

In the following $\vartheta = -\left[\frac{\partial_\xi \rho}{\partial_z \rho}\right] = const$

$\longrightarrow$ Local computation of neutral directions (cf Sasha's talk)

Discretization of the Redi tensor $\mathcal{D} = \frac{1}{\xi}\left[F^{(\xi)}_{i+\frac{1}{2},k} - F^{(\xi)}_{i-\frac{1}{2},k}\right]$



- **TRIADS Scheme** (Griffies *et al.*, 1998) $\Rightarrow$ Self-adjoint property

$$F^{(\xi)}_{i+\frac{1}{2},k} = K_h\left[\frac{q_{i+1,k}-q_{i,k}}{\Delta\xi} - \frac{1}{4}\frac{\rho_{i+1,k}-\rho_{i,k}}{\Delta\xi}\left\{\frac{q_{i,k}-q_{i,k-1}}{\rho_{i,k}-\rho_{i,k-1}} + \frac{q_{i+1,k+1}-q_{i+1,k}}{\rho_{i+1,k+1}-\rho_{i+1,k}} + \frac{q_{i,k+1}-q_{i,k}}{\rho_{i,k+1}-\rho_{i,k}} + \frac{q_{i+1,k}-q_{i+1,k-1}}{\rho_{i+1,k}-\rho_{i+1,k-1}}\right\}\right]$$

- **SWITCH Scheme** (Beckers *et al.*, 2000, Shchepetkin, 1998 (unpublished))

$$F^{(\xi)}_{i+\frac{1}{2},k} = K_h\left[\frac{q_{i+1,k}-q_{i,k}}{\Delta\xi} - \frac{1}{2}\frac{\max(\rho_{i+1,k}-\rho_{i,k},0)}{\Delta\xi}\left\{\frac{q_{i,k}-q_{i,k-1}}{\rho_{i,k}-\rho_{i,k-1}} + \frac{q_{i+1,k+1}-q_{i+1,k}}{\rho_{i+1,k+1}-\rho_{i+1,k}}\right\} - \frac{1}{2}\frac{\min(\rho_{i+1,k}-\rho_{i,k},0)}{\Delta\xi}\left\{\frac{q_{i,k+1}-q_{i,k}}{\rho_{i,k+1}-\rho_{i,k}} + \frac{q_{i+1,k}-q_{i+1,k-1}}{\rho_{i+1,k}-\rho_{i+1,k-1}}\right\}\right]$$

Stability and accuracy of the isopycnal-mixing equation

- **Truncation error**

$$\varepsilon = \mathcal{N}(\mathcal{D}) - \mathcal{D} = \frac{\Delta\xi^2}{12}\partial_\xi^4 q + \vartheta^2 \frac{\Delta z^3_{k+\frac{1}{2}} + \Delta z^3_{k-\frac{1}{2}}}{24\Delta z_k}\partial_z^4 q + \vartheta \frac{(\Delta z_{k+\frac{1}{2}} + \Delta z_{k-\frac{1}{2}})\Delta\xi}{4}\partial_\xi^2\partial_z^2 q$$

$$+2\vartheta\left[\frac{\Delta z^2_{k+\frac{1}{2}} + \Delta z^2_{k-\frac{1}{2}}}{12}\partial_\xi\partial_z^3 q + \frac{\Delta\xi^2}{6}\partial_\xi^3\partial_z q\right] + \mathcal{O}(\Delta z^3, \Delta\xi^3, \Delta\xi^2\Delta z, \Delta\xi\Delta z^2)$$

$\Rightarrow$ the rotation adds terms of dispersive nature (SWITH-TRIADS $\Rightarrow$ green diffusive term)

- **Stability**

with Euler time stepping :

$$q^{n+1} - q^n = \Delta t K_h \left(\partial_\xi^2 q^n + 2\vartheta\partial_\xi\partial_z q^n + \vartheta^2\left[(1-\alpha)\partial_z^2 q^n + \alpha\partial_z^2 q^{n+1}\right]\right)$$

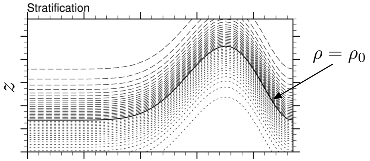
Stability constraint :

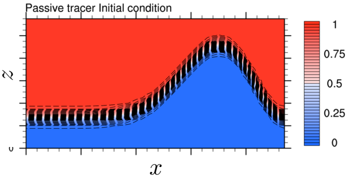
$$\alpha = 0 \quad \rightarrow \frac{K_h\Delta t}{\Delta\xi^2}\left(1 + \frac{\Delta z^2}{\Delta\xi^2}\vartheta^2\right) \le \frac{1}{2} \quad ; \quad \alpha = 1 \quad \rightarrow \frac{K_h\Delta t}{\Delta\xi^2} \le \frac{1}{2}$$

For the full tensor and $\alpha = 0 \quad \rightarrow \dfrac{K_h\Delta t}{\Delta\xi^2} \le \dfrac{1}{2}$.

Testcase

- Isopycnal diffusion should not affect a tracer homogeneous along isopycnals





- Diffusion equation discretized in (x,z)
- The diffusion is rotated along isopycnals

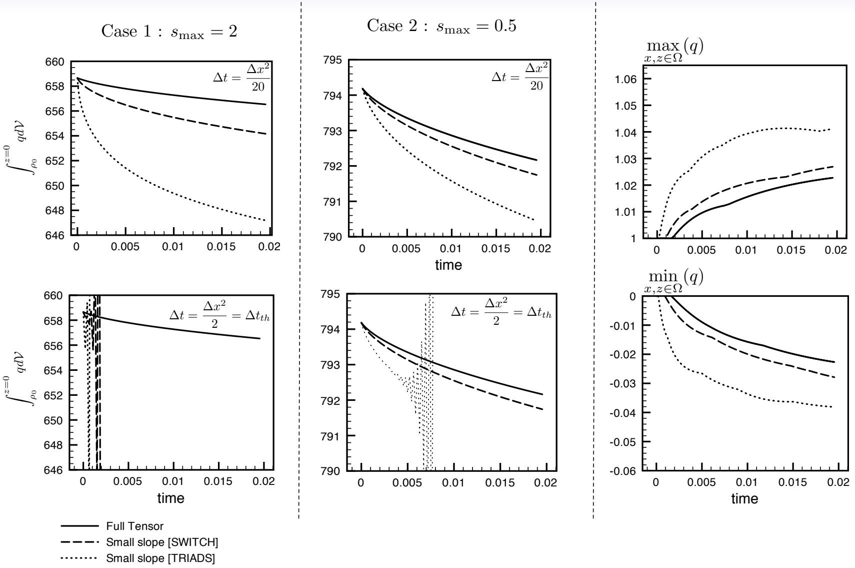
Two stratifications :

1. Large slope $s_{\max} = 2$ $(\approx 48^o)$
2. Moderate slope $s_{\max} = 0.5$ $(\approx 25^o)$

Two discretizations of the tensor :

1. Small slope approximation
2. Full tensor

Numerical stability, overshots and cross-isopycnal flux



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A first solution (Marchesiello *et. al*, 2009)

RSUP3

$$\begin{cases} q^{n+1,\star} & = & q^{n-1} - 2\Delta t \left[u\partial_x q^n\right]_{\mathrm{C4}} \\ q^{n+1/2} & = & \frac{5}{12} q^{n+1,\star} + \frac{2}{3} q^n - \frac{1}{12} q^{n-1} \\ q^{n+1} & = & q^n - \Delta t \left[u\partial_x q^{n+1/2}\right]_{\mathrm{C4}} + \Delta t \mu(\Delta x, u) \Delta^2 (\alpha_1 q^{n+1/2} + \alpha_2 q^n) \\ & & \text{with} \qquad \Delta = (\partial_x + \vartheta \partial_z)^2 \end{cases}$$

However let's look if we can find

- $\longrightarrow$ either a simpler alternative
- $\longrightarrow$ or an alternative with a wider stability range

- Idea : take advantage of the form of the truncation error of the first-order upstream scheme (i.e. an anisotropic Laplacian in 2D)

Scheme 1

$$\begin{cases} q^{n+1,\star} & = & q^{n-1} - 2\Delta t \left[u\partial_x q^n\right]_{\mathrm{C4}} \\ q^{n+1/2} & = & \dfrac{5}{12} q^{n+1,\star} + \dfrac{2}{3} q^n - \dfrac{1}{12} q^{n-1} \\ q^{n+1} & = & q^n - \Delta t \left[u\partial_x q^{n+1/2}\right]_{\mathrm{C4}} \\ q^\star & = & q^n - \Delta t \left[u\partial_x q^n\right]_{\mathrm{UP1}} \\ q^{n+1} & = & q^{n+1} - \mu(\Delta x)\Delta(q^\star - q^{n+1}) \qquad \Delta = \partial_x^2 \end{cases}$$

Scheme 1 rotated anisotropic "implicit" laplacian

$$
\begin{cases}
q^{n+1,\star} = q^{n-1} - 2\Delta t \left[u\partial_x q^n\right]_{\mathrm{C4}} \\
q^{n+1/2} = \dfrac{5}{12} q^{n+1,\star} + \dfrac{2}{3} q^n - \dfrac{1}{12} q^{n-1} \\
q^{n+1} = q^n - \Delta t \left[u\partial_x q^{n+1/2}\right]_{\mathrm{C4}} \\
q^\star = q^n - \Delta t \left[u\partial_x q^n\right]_{\mathrm{UP1}} \\
q^{n+1} = q^{n+1} - \mu(\Delta x)\Delta(q^\star - q^{n+1}) \qquad \Delta = (\partial_x + \vartheta\partial_z)^2
\end{cases}
$$

- extremely simple to implement and computationally efficient
- does not correspond exactly to a rotated biharmonic

  $\text{Rotated biharmonic} = \text{Rotation of a non}-\text{rotated Laplacian} + \mathcal{O}(\vartheta^3, \vartheta^4)$

  Perfectly valid for small slopes and thus for rotation along geopotentials
- aims at reducing spurious mixing to an acceptable level but not at suppressing it

Scheme 2

$$
\left\{
\begin{array}{lcll}
q^{n+1,\star} & = & q^{n-1} - 2\Delta t \left[u\partial_x q^n\right]_{\mathrm{C4}} & \\
q^{n+1,\star\star} & = & q^{n+1,\star} + 2\Delta t \Delta(q^n) & \Delta = \partial_x^2 \\
q^{n+1/2} & = & \frac{5}{12} q^{n+1,\star} + \frac{2}{3} q^n - \frac{1}{12} q^{n-1} & \\
q^{n+1} & = & q^n - \Delta t \left[u\partial_x q^{n+1/2}\right]_{\mathrm{C4}} & \\
q^{n+1} & = & q^{n+1} - \mu(\Delta x, u)\Delta(q^{n+1,\star\star} - q^{n+1}) & \Delta = \partial_x^2
\end{array}
\right.
$$

- We split the biharmonic in a first laplacian during the predictor and a second in the corrector

Scheme 2 rotated

$$\begin{cases} q^{n+1,\star} & = & q^{n-1} - 2\Delta t \left[u \partial_x q^n \right]_{\mathrm{C4}} & \\ q^{n+1,\star\star} & = & q^{n+1,\star} + 2\Delta t \Delta(q^n) & \Delta = (\partial_x + \vartheta \partial_z)^2 \\ q^{n+1/2} & = & \frac{5}{12} q^{n+1,\star} + \frac{2}{3} q^n - \frac{1}{12} q^{n-1} & \\ q^{n+1} & = & q^n - \Delta t \left[u \partial_x q^{n+1/2} \right]_{\mathrm{C4}} & \\ q^{n+1} & = & q^{n+1} - \mu(\Delta x, u) \Delta(q^{n+1,\star\star} - q^{n+1}) & \Delta = (\partial_x + \vartheta \partial_z)^2 \end{cases}$$

- Two rotated laplacians are simpler than a rotated biharmonic
- makes it possible to derive an unconditionally stable scheme

Scheme $2^\star$ (for the vertical component of the tensor)

$$\begin{cases} q^{n+1,\star} & = & q^{n-1} - 2\Delta t \left[w \partial_z q^n \right] & \\ q^{n+1,\star\star} & = & q^{n+1,\star} + 2\Delta t \Delta(q^{n+1,\star\star}) & \Delta = \partial_z^2 \\ q^{n+1/2} & = & \frac{5}{12} q^{n+1,\star} + \frac{2}{3} q^n - \frac{1}{12} q^{n-1} & \\ q^{n+1} & = & q^n - \Delta t \left[w \partial_z q^{n+1/2} \right] + \mu \Delta(q^{n+1} - q^{n+1,\star\star}) & \Delta = \partial_z^2 \end{cases}$$

- resolution of one (possibly two) tridiagonal systems at each time steps
- this results into an implicit-in-time biharmonic in the vertical direction
- however no need to decrease the time step
- but increase of the complexity of the code

Comments on stability

- **Scheme** 1

$$\mathcal{R}(q) = \partial_\xi^4 q + \vartheta^2 \partial_z^2 \partial_\xi^2 q + 2\vartheta \partial_\xi^3 \partial_z q$$

$\Rightarrow$ Should not require any additional limiters for stability.

- **Scheme** 2

$$\mathcal{R}(q) = \partial_\xi^4 q + 6\vartheta^2 \partial_z^2 \partial_\xi^2 q + 4\vartheta \partial_\xi^3 \partial_z q + 4\vartheta^3 \partial_\xi \partial_z^3 q + \vartheta^4 \partial_z^4 q$$

$\Rightarrow$ Need to incorparate *ad-hoc* prescriptions in order to keep stability for steep slopes.

- **Scheme** $2^\star$: unconditionally stable

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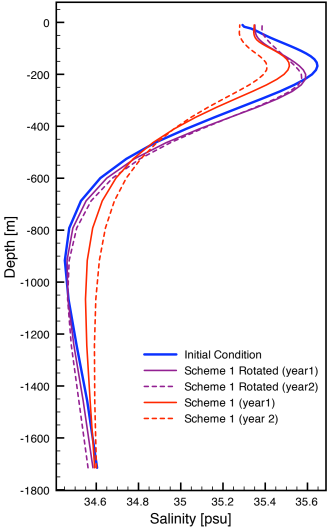
South West Pacific



ROMS-UCLA : Southwest Pacific Region

$\Delta x = 20$km

1. Scheme 1 (no rotation)
2. Scheme 1 rotated along geopotentials

For this particular testcase no need to decrease the time step

$$\Delta t = \Delta t_{\text{rotated}} = 2160s$$

(N=40 vertical levels)

$\Rightarrow$ reduction of the error by 60%

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en route to a physical parameterization of diapycnal mixing for the ocean interior

By restoring (part of) the adiabaticity in the ocean interior we open the door to a closure scheme for diapycnal mixing

$$\mathbf{K}^{\mathrm{small}} = K_h \begin{pmatrix} 1 & 0 & \vartheta_x \\ 0 & 1 & \vartheta_y \\ \vartheta_x & \vartheta_y & \varepsilon + \vartheta_y^2 + \vartheta_x^2 \end{pmatrix}$$

$$\varepsilon = \frac{\mathbf{K_D}}{K_h}$$

- The formalism of the tensor allows for the definition of a diapycnal diffusivity K_D
- *e.g.* connection between internal waves and diapycnal mixing

Putting the Pieces of the Puzzle Together : the missing piece !